Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(04)00029-2
J. Appl. Maths Mechs, Vol. 68, No. 2, pp. 199–205, 2004

0021-8928/$—see front matter

METELITSYN'S INEQUALITY AND STABILITY CRITERIA FOR MECHANICAL SYSTEMS†

W. KLIEM and A. P. SEYRANIAN

Copenhagen and Moscow

e-mail: w.kliem@mat.dtu.dk; seyaran@imec.msu.ru

(*Received* 16 *August* 2002)

Criteria of asymptotic stability for general linear mechanical systems are investigated. It is shown that the inequality first derived by Metelitsyn (1952) is a sufficient but not necessary condition for asymptotic stability. We argue that this inequality is of little use in applications. Metelitsyn's theorems based on his inequality as well as critical comments in the literature on these theorems are analysed. Practical sufficient stability criteria are obtained in terms of extreme eigenvalues of the system matrices. This analysis is of special value for rotor systems in a complex setting which is demonstrated by three examples. © 2004 Elsevier Ltd. All rights reserved.

1. INTRODUCTION

It is surprising that Metelitsyn's 50-years-old stability criterion and stability theorems for linear non-conservative systems [1, 2] still cause comments and confusion. Similar stability conditions were derived and dealt with by Frik [3] and Huseyin [4]. The latest papers on this subject are by Seyranian [5], Kliem *et al.* [6], Merkin [7], and Zhbanov and Zhuravlev [8].

The subject of investigation is the stability of linear systems of the form

$$A\ddot{q} + (B+G)\dot{q} + (C+N)q = 0 \tag{1.1}$$

which are general non-conservative models in mechanics. Here, A, B, G, C, and N are real $m \times m$ matrices. The mass matrix A is assumed to be symmetric and positive definite, $A^T = A > 0$. Damping is characterized by the symmetric matrix B, and gyroscopic matrix G is skew-symmetric, $G = -G^T$. The potential forces are described by the symmetric matrix C, and the non-conservative positional forces by the skew-symmetric matrix N. Finally, the vector q represents the generalized coordinates of the system.

Assuming solutions of the form $q = he^{\lambda t}$, the stability of system (1.1) can be completely understood in terms of the algebraic eigenvalue problem

$$[\lambda^2 A + \lambda(B+G) + C + N]h = 0, h \neq 0 \tag{1.2}$$

the eigenvalues λ are the roots of the characteristic polynomial of degree $2m$, $\det[\lambda^2 A + \lambda(B + G) + C + N]$, and if all the eigenvalues have negative real parts, system (1.1) is asymptotically stable. An investigation of the real parts of the eigenvalues with help of the Routh–Hurwitz criterion is very cumbersome and moreover, in this approach the properties of the system matrices having physical meaning play no role. Therefore, Metelitsyn and others went alternative ways in investigating stability.

Since we do not entirely agree with the recent contributions [7, 8] on Metelitsyn's theory we present this paper with the following aims:

1. To give a short alternative derivation of Metelitsyn's inequality, his so-called stability condition.
2. To state the reason why this stability condition is sufficient but not necessary for asymptotic stability.
3. To explain why the stability condition in the form given by Metelitsyn is of little use in practice and comment on his seven stability theorems.
4. To show how Metelitsyn's inequality leads to a practical stability condition expressed by extreme eigenvalues of the system matrices.
5. To point out the special advantage of this applicable condition for systems with complex symmetric matrices resulting in stability criteria for rotor systems.
6. Finally, to give examples showing that the results gained from this applicable condition lead to qualitatively correct stability statements but not always to exact stability boundaries.

†*Prikl. Mat. Mekh.* Vol. 68, No. 2, pp. 225–233, 2004.

2. DERIVATION OF METELITSYN'S INEQUALITY

Metelitsyn pre-multiplied Eq. (1.2) by the conjugate transposed eigenvector h^* and obtained the equation (one equation for each eigenvalue λ)

$$T\lambda^2 + (D + i\Gamma)\lambda + V + iE = 0 \tag{2.1}$$

with the coefficients

$$T = h^*Ah, \quad D = h^*Bh, \quad i\Gamma = h^*Gh, \quad V = h^*Ch, \quad iE = h^*Nh \tag{2.2}$$

where T, D, Γ, V, and E are real quantities. Additionally we assume normalized eigenvectors, $h^*h = 1$.

Metelitsyn's famous inequality now follows from claiming that *both* roots of (2.1), considered as a quadratic equation in λ, have negative real parts. Instead of an elementary but lengthy computation of these roots we will use a theorem by Bilharz and Schur [9]. This theorem states that both roots of Eq. (2.1) have negative real parts if and only if the two determinants satisfy the relations

$$\begin{vmatrix} T & \Gamma \\ 0 & D \end{vmatrix} > 0, \quad \begin{vmatrix} T & \Gamma & -V & 0 \\ 0 & D & E & 0 \\ 0 & T & \Gamma & -V \\ 0 & 0 & D & E \end{vmatrix} > 0 \tag{2.3}$$

Since the matrix $A > 0$, the quantity $T > 0$, and then (2.3) is equivalent to the two conditions

$$D > 0 \tag{2.4}$$

$$TE^2 - \Gamma DE < D^2V \tag{2.5}$$

Metelitsyn [1, 2] was the first to derive inequality (2.5) – by assuming condition (2.4) – and he called it "the condition of (asymptotic) stability of non-conservative systems" of the form (1.1).

3. METELITSYN'S INEQUALITY IS SUFFICIENT BUT NOT NECESSARY FOR ASYMPTOTIC STABILITY

Notice that the eigenvalue λ is one of the two roots of Eq. (2.1), the other root need not be an eigenvalue. This important fact was pointed out in [5] and then repeated in [8]. But it was not recognized by Metelitsyn [1, 2] nor in [4], and is not mentioned in [7]. Actually, it is more an exception than a rule that both roots are eigenvalues. The following can be shown following a private communication by C. Pommer.

Consider λ_r as an eigenvalue of (1.2) with eigenvector h_r and corresponding polynomial (2.1). Let (2.1) have two different roots λ_r and λ_s. Then λ_s is also an eigenvalue of the system if it has $\bar{h}_r$ as a left eigenvector, i.e. $h_r^*[\lambda_s^2 A + \lambda_s(B + G) + C + N] = 0$. An example for this situation is a weakly damped system with $G = 0$ and $N = 0$, since we can choose $\lambda_s = \bar{\lambda}_r$.

Metelitsyn's mistake, also made in [4], was to believe that both roots of (2.1) are always eigenvalues of problem (1.2). This mistake led to the incorrect conclusion that inequality (2.5) is a *necessary* and sufficient condition for stability. Obviously, inequalities (2.4) and (2.5) are *sufficient for asymptotic stability, but not necessary*. This can be easily demonstrated by Merkin's example [7].

Let system (1.1) be given by

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}\ddot{q} + \left(\begin{Vmatrix} 5.8186 & 0 \\ 0 & 0.1814 \end{Vmatrix} + \begin{Vmatrix} 0 & 3.6667 \\ -3.6667 & 0 \end{Vmatrix}\right)\dot{q} + \\ + \left(\begin{Vmatrix} -0.5 & 0 \\ 0 & -0.5 \end{Vmatrix} + \begin{Vmatrix} 0 & 2.25 \\ -2.25 & 0 \end{Vmatrix}\right)q = 0 \tag{3.1}$$

The eigenvalues are $\lambda_{1,2} = -1 \pm 0.5\,i$ and $\lambda_{3,4} = -2 \pm 0.5\,i$, and therefore system (3.1) is asymptotically stable. Computing the corresponding eigenvectors h, the coefficients (2.2) of quadratic equation (2.1) can be determined. The roots of this equation (one equation for each eigenvalue) are of course the four eigenvalues $\lambda_{1,2}$ and $\lambda_{3,4}$, but additionally also $0.0625 \pm 0.875\,i$ and $0.1786 \pm 0.2857\,i$. These roots have positive real parts, and therefore – in spite of system (3.1) being asymptotically stable – Metelitsyn's inequality (2.5) is not satisfied, since it requires that both roots of (2.1) should have negative real parts.

4. WHY METELITSYN'S INEQUALITY IS OF LITTLE USE IN PRACTICE AND COMMENTS ON HIS THEOREMS

If we wish to investigate the possibility of asymptotic stability for a given system by checking inequalities (2.4) and (2.5) as sufficient conditions, we face the following problem.

Metelitsyn required condition (2.5) to be satisfied when formulating several stability theorems. But the eigenvectors h – which are used for coefficients (2.2) and for inequalities (2.4) and (2.5) – are *unknown* (this was pointed out, for example, in [3, 7]). They can only be determined by solving eigenvalue problem (1.2), and then the stability analysis would be complete. So we disagree with [8] and state that Metelitsyn's inequality (2.5) cannot be checked in the given form without computing eigenvectors (i.e. also eigenvalues) of (1.2).

However, as shown in [5], Metelitsyn's inequality (2.5) leads to the third Thomson–Tait–Chetayev theorem, see [10]: a statically stable system ($C > 0$, which implies $V > 0$) becomes asymptotically stable if arbitrary gyroscopic forces and dissipative forces with complete dissipation ($D > 0$) are added. Indeed, in the case when $N = 0$ (i.e. $E = 0$), inequality (7) reduces to $D^2V > 0$, guaranteeing asymptotic stability.

In the general case, based on inequalities (2.4), (2.5), Metelitsyn [1, 2] formulated seven theorems, (see also [7]). Two of them are also presented in [11]. Our brief comments on these theorems are the following.

Theorems 1 and 2, dealing with systems with only positional forces (without dissipative and gyroscopic forces), cannot be deduced from (2.5), since without dissipation ($D = 0$) condition (2.4) is violated, and asymptotic stability for such systems cannot be achieved.

The counter-examples for Theorems 3 and 4 are systems with odd number of degrees of freedom, which can never be stabilized by dissipative and gyroscopic forces, because the free term in the characteristic equation for systems with only non-conservative positional forces ($C = 0$), or for statically unstable systems with $C < 0$, is zero or a negative number, respectively, resulting in violation of the Routh–Hurwitz criterion. This was pointed out by Merkin in [7, 10]. Consequently for such systems inequality (2.5) can never be satisfied. For example, let system (1.1) be of dimension 3 with the matrices

$$C = \begin{Vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{Vmatrix}, \quad N = \begin{Vmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{Vmatrix}$$

Then the free term of the characteristic equation for the system $\det[C + N] = abc + b\beta^2 + a\gamma^2 + c\alpha^2$ is zero for $a = b = c = 0$, ($C = 0$), or is a negative number for $a < 0, b < 0, c < 0$, ($C < 0$) and arbitrary α, β, γ.

Theorem 5 and 6 are not related to inequalities (2.4) and (2.5) (see also [8]). The counter-example for Theorem 7 given in [7] apparently shows that it is wrong.

Therefore, we disagree with some of the comments in [8] concerning the practical use of inequality (2.5) and the validity of these seven theorems. We conclude that Metelitsyn failed in trying to use inequalities (2.4), (2.5) to formulate qualitative stability conditions for system (1.1). Nevertheless, we will show in the next sections that Metelitsyn's inequality (2.5) helps to establish applicable sufficient stability conditions.

5. A PRACTICAL SUFFICIENT STABILITY CONDITION

The field of values of a real or complex $m \times m$ matrix M is the set of complex numbers x^*Mx, where x ranges over all complex m-dimensional vectors that are normalized, $x^*x = 1$. The quantities defined in (2.2) are situated in the field of values for the respective matrices. Hermitian matrices like A, B, and C (real symmetric in our case) have only real eigenvalue. Their field of values is real as well as limited by minimum and maximum eigenvalues of the matrices, respectively. (see e.g. [12]). The quantities T, D, and V from (2.2), known as Rayleigh quotients, are therefore also limited by minimum and maximum eigenvalues of the matrices A, B, and C, respectively, although we don't know the eigenvectors h. We emphasize that these limits depend only on the system matrices

$$a_1 = \lambda_{\min}(A) \le T \le \lambda_{\max}(A) = a_2, \quad b_1 = \lambda_{\min}(B) \le D \le \lambda_{\max}(B) = b_2$$
$$c_1 = \lambda_{\min}(C) \le V \le \lambda_{\max}(C) = c_2 \tag{5.1}$$

The skew-Hermitian matrices G and N possess only purely imaginary eigenvalues and their field of values is imaginary also. In our case the matrices G and N are real skew-symmetric and iG and iN are Hermitian. Therefore the fields of values of G and N are limited by the eigenvalues of maximum absolute value $-ig$ and ig, and $-in$ and in, respectively, where $g = |\lambda(G)|_{\max}$ and $n = |\lambda(N)|_{\max}$. So we have

$$-g \le \Gamma \le g, \quad -n \le E \le n \tag{5.2}$$

If we assume

$$A > 0, \quad B > 0, \quad C > 0 \tag{5.3}$$

then it is easy to see, with the help of (5.2) and (5.3), then (2.5), rewritten in the form $D(DV + \Gamma E) - TE^2 > 0$, is satisfied for arbitrary vectors h if

$$b_1(b_1c_1 - gn) - a_2n^2 > 0 \tag{5.4}$$

Here we took the smallest value of the first term and the largest values of the second and third terms of the inequality.

Under assumption (5.3), inequality (5.4) is a *practical sufficient condition* for asymptotic stability of system (1.1), which can be checked knowing only the extreme eigenvalue of the system matrices A, B, C, G, and N. In [3, 5, 6] similar conditions were derived, however such simple but important considerations do not occur [1, 2, 7, 8] on the subject.

In [3] inequality (5.4) was solved for b_1 resulting in the assertion: system (1.1) is asymptotically stable if, in addition to (5.3), the damping is sufficiently large

$$b_1 > n(g + \sqrt{g^2 + 4a_2c_1})/(2c_1) \tag{5.5}$$

From (5.4) we can also deduce the following stability statement: system (1.1), under assumption (5.3), can be stabilized by sufficiently large dissipative and/or potential forces. This follows, by making the term $b_1^2c_1$ sufficiently large, implying (5.4) is satisfied. This result was reported in [5]. Another consequence of inequality (5.4) is that a statically stable system with complete dissipation (assumption (5.3)) cannot be destabilized by adding rather small gyroscopic and positional non-conservative forces.

6. THE STABILITY OF ROTOR SYSTEMS

Free lateral vibrations of a large class of rotor systems and centrifuges, where the rotating elements are symmetrical about the rotor axis and the bearings are isotropic, can be described by non-conservative systems, see e.g. [6, 13, 14]

$$A_1\ddot{q} + (B_1 + G_1)\dot{q} + (C_1 + N_1)q = 0 \tag{6.1}$$

with the block matrices

$$A_1 = \begin{Vmatrix} A & 0 \\ 0 & A \end{Vmatrix}, \quad B_1 = \begin{Vmatrix} B & 0 \\ 0 & B \end{Vmatrix}, \quad G_1 = \begin{Vmatrix} 0 & G \\ -G & 0 \end{Vmatrix}, \quad C_1 = \begin{Vmatrix} C & 0 \\ 0 & C \end{Vmatrix}, \quad N_1 = \begin{Vmatrix} 0 & N \\ -N & 0 \end{Vmatrix}$$

Here A, B, G, C, and N are *all symmetric*. Moreover, A is positive definite (>0) while B, G, and N are positive semi-definite ($\geqslant 0$).

A convenient rewriting of system (6.1) results in the complex system.

$$A\ddot{z} + (B + iG)\dot{z} + (C + iN)z = 0; \quad z = q_1 - iq_2, \quad q = \begin{Vmatrix} q_1 \\ q_2 \end{Vmatrix} \tag{6.2}$$

where $z = q_1 - iq_2$ and $q = \begin{bmatrix} q_1 \\ \hline q_2 \end{bmatrix}$.

If λ is an eigenvalue of the complex system (6.2), then λ and the complex conjugate $\bar{\lambda}$ are eigenvalues of the real system (6.1). One of the advantages of the complex setting is that the dimension of the matrices is halved. Notice that in Eq. (6.2) the system matrices $B + iG$ and $C + iN$ are complex symmetric and are divided into their Hermitian parts $B \geqslant 0$ and $C > 0$ and skew-Hermitian parts iG and iN. Since $G \geqslant 0$ and $N \geqslant 0$, another advantage of form (6.2) is that the quantities Γ and E in Metelitsyn's inequality are now limited by

$$0 \leq g_1 \leq \Gamma \leq g_2, \quad 0 \leq n_1 \leq E \leq n_2 \tag{6.3}$$

where g_1, n_1 and g_2, n_2 are the smallest and largest eigenvalues of the matrices G and N, respectively. Then inequality (5.4) is improved to be

$$b_1(b_1c_1 + g_1n_1) - a_2n_2^2 > 0 \tag{6.4}$$

Damping is normally split into external damping B_e and internal damping B_i, $B = B_e + B_i$, and both G and N are linear matrix functions of the angular velocity Ω of the rotor system, see [13]

$$G = \Omega G_0, \quad N = \Omega B_i \tag{6.5}$$

where $G_0 \geqslant 0$ is a constant symmetric matrix. This structure of N assumes coordinates q with respect to an inertial frame. Using a frame rotating with Ω, external damping B_e will appear in N, and C will be dependent on Ω (see example 1, below).

Let γ_1 be the smallest eigenvalue of G_0, and d_1 and d_2 the smallest and largest eigenvalues of B_i, respectively. Using (6.5), we substitute these quantities into (6.4) and obtain the following estimate for Ω ensuring stability

$$\Omega^2(a_2d_2^2 - d_1\gamma_1b_1) < c_1b_1^2 \tag{6.6}$$

But we can do even better. Metelitsyn's inequality (2.5) in the form $E(TE - \Gamma D) < D^2V$ is satisfied if

$$\Omega^2 d_2(a_2d_2 - \gamma_1b_1) < c_1b_1^2 \tag{6.7}$$

which is advantageous compared with (6.6). We repeat, a_2 and d_2 are the largest eigenvalues of A and B_i while b_1, c_1, and γ_1 are the smallest eigenvalues of B, C, and G_0, respectively. Together with the assumption $B > 0$ inequality (6.7) represents an effective *sufficient criterion* for asymptotic stability of rotor system (6.2) and enables us to determine an estimate of the angular velocity Ω permissible for stability (see Examples 1 and 2 below).

If $c_1 \geqslant 0$ and $a_2d_2 < \gamma_1b_1$, then (6.7) is satisfied for all values of Ω (gyroscopic stabilization). However, (6.7) can be satisfied even for $c_1 < 0$ (the statically unstable case when C is not positive definite) (see Example 1).

7. EXAMPLES

Example 1

The simplest rotor consists of a massless shaft of circular cross section with stiffness $k > 0$, rotating with constant angular velocity Ω and carrying a single disk of mass m. External and internal damping are denoted by $d_e > 0$ and $d_i > 0$, respectively. With respect to an inertial frame, the equations of motion for the centre of mass of the disk moving in a plane perpendicular to the shaft are (see [15])

$$\begin{Vmatrix} m & 0 \\ 0 & m \end{Vmatrix} \ddot{q} + \begin{Vmatrix} d_e + d_i & 0 \\ 0 & d_e + d_i \end{Vmatrix} \dot{q} + \left\{ \begin{Vmatrix} k & 0 \\ 0 & k \end{Vmatrix} + \begin{Vmatrix} 0 & d\Omega \\ -d_i\Omega & 0 \end{Vmatrix} \right\} q = 0 \tag{7.1}$$

In the complex setting (6.2) the equation of motion has the form

$$m\ddot{z} + (d_e + d_i)\dot{z} + (k + id_i\Omega)z = 0 \tag{7.2}$$

The smallest and largest eigenvalues of the system matrices are identical

$$a_1 = a_2 = m, \quad b_1 = b_2 = d_e + d_i, \quad c_1 = c_2 = k, \quad d_1 = d_2 = d_i$$

For system (7.1) inequality (5.4) results in

$$\Omega^2 < k(d_e + d_i)^2/(md_i^2) \tag{7.3}$$

Inequality (6.7) applied to the complex equation (7.2) gives the same result (7.3), which actually determines the correct stability limit of the system (see [15]).

If we transform the equations of motion (7.1) to a reference system spinning with the angular velocity Ω of the rotor, we get

$$\begin{Vmatrix} m & 0 \\ 0 & m \end{Vmatrix} \ddot{q} + \left\{ \begin{Vmatrix} d_e + d_i & 0 \\ 0 & d_e + d_i \end{Vmatrix} + \begin{Vmatrix} 0 & 2m\Omega \\ -2m\Omega & 0 \end{Vmatrix} \right\} \dot{q} + \\ + \left\{ \begin{Vmatrix} k - m\Omega^2 & 0 \\ 0 & k - m\Omega^2 \end{Vmatrix} + \begin{Vmatrix} 0 & d_e\Omega \\ -d_e\Omega & 0 \end{Vmatrix} \right\} q = 0 \tag{7.4}$$

or in a complex setting

$$m\ddot{\varsigma} + (d_e + d_i + i2m\Omega)\dot{\varsigma} + (k - m\Omega^2 + id_e\Omega)\varsigma = 0 \tag{7.5}$$

In (7.4) and (7.5) we recognize Coriolis and centrifugal terms. Inequality (5.4) cannot be applied to (7.4) since C is not necessarily positive definite (see assumption (5.3)). But inequality (6.7) can be used for Eq. (7.5) resulting in

$$\Omega^2 d_e(md_e - 2m(d_e + d_i)) < (k - m\Omega^2)(d_e + d_i)^2 \tag{7.6}$$

which again leads to stability limit (7.3).

Example 2
It is well known that asymmetrical steam flow in turbines leads to asymmetrical forces on the rotor blades and can therefore be the source of instabilities. A simple model of this situation is considered in [16]. With respect to an inertial frame the equation of motion in a complex setting has the form

$$m\ddot{z} + (d_e + d_i)\dot{z} + (k + i(d_i\Omega + k_s))z = 0 \tag{7.7}$$

here m, d_e, d_i, and k denote the same coefficients as in example 1, and k_s represents the steam flow. Since (6.5) was assumed to obtain (6.6) and (6.7), these inequalities cannot be applied in our case. Therefore we use inequality (6.4) with

$$a_2 = m, \ b_1 = d_e + d_i, \ c_1 = k, \quad n_1 = n_2 = \Omega d_i + k_s$$

resulting in stability for

$$k_s < (d_e + d_i)\sqrt{k/m} - \Omega d_i \tag{7.8}$$

Inequality (7.8) gives the correct stability limit, which was determined in [16] by computing the eigenvalues of the system.

Example 3
Consider the rotor from example 1 but now additionally with mass m_b, damping d_b and stiffness k_b in the bearings (see Fig. 1). A linear model is described by a complex 2×2 system (or a real 4×4 system) (see [6])

$$\begin{Vmatrix} m & 0 \\ 0 & m_b \end{Vmatrix} \ddot{z} + \begin{Vmatrix} d_e + d_i & -d_i \\ -d_i & d_b + d_i \end{Vmatrix} \dot{z} + \left\{ \begin{Vmatrix} k & -k \\ -k & k + k_b \end{Vmatrix} + i \begin{bmatrix} d_i\Omega & -d_i\Omega \\ -d_i\Omega & d_i\Omega \end{bmatrix} \right\} z = 0 \tag{7.9}$$

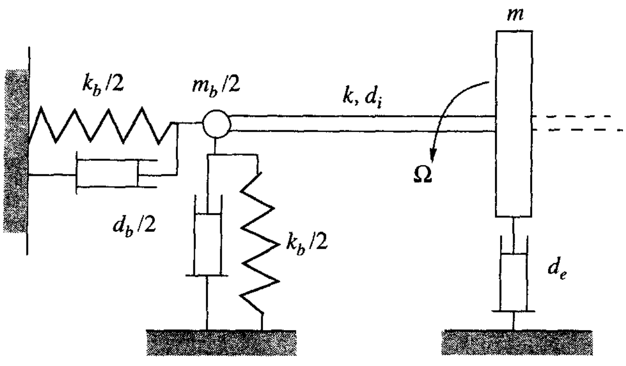

Fig. 1

Omitting all dimensions for simplicity, we consider the specific case

$$m = 1, \quad d_i = 1, \quad d_e = 5, \quad d_b = 10, \quad k = 100, \quad k_b = 400$$

and different bearing masses $m_b \leqslant 1$. Then

$$a_2 = 1, \quad d_2 = 2, \quad b_1 = 5.8, \quad c_1 = 76.4, \quad \gamma_1 = 0$$

such that inequality (6.7) results in $\Omega < 25.3$ for stability. But for $m_b = 1$ the correct stability limit is $\Omega_{crit} = 110$, for $m_b = 0.5$ we have $\Omega_{crit} = 97.7$ and for $m_b = 0.1$ we have $\Omega_{crit} = 92$. These examples show that inequality (6.7) normally implies pessimistic stability boundaries. This effect may increase as the dimension of the system increase.

The examples show that for simple systems inequalities (5.4), (6.4), and (6.7) can reveal the true stability boundary, but for more complicated systems with higher dimensions can give qualitatively correct but not good quantitative results.

We wish to thank Christian Pommer for fruitful discussions and express our appreciation of the interest to the late Academician A. Yu. Ishlinskii in this research.

REFERENCES

1. METELITSYN, I. I., The problem of gyroscopic stabilization. *Dokl. Akad. Nauk SSSR*, 1952, **86**, 1, 31–34.
2. METELITSYN, I. I., *Collected Papers. Theory of the Gyroscope. Theory of Stability*. Nauka, Moscow, 1977.
3. FRIK, M., Zur Stabilität nichtkonservativer linearer Systeme. *ZAMM*, 1972, **52**, T47–T49.
4. HUSEYIN, K., *Vibrations and Stability of Multiple Parameter Systems*. Alphen aan den Rijn: Sijthoff & Noordhoff, 1978.
5. SEYRANIAN, A. P., On Metelitsyn's theorems. *Izv. Ross. Akad. Nauk. MTT*, 1994, **29**, 3, 39–43.
6. KLEIM, W., POMMER, C. and STOUSTRUP, J., Stability of rotor systems: A complex modelling approach. *Z. angew. Math. Phys. (ZAMP)*, 1998, **49**, 644–655.
7. MERKIN, D. R., Metelitsyn's methods and theorems. *J. Appl. Maths Mechs*, 2001, **65**, 3, 519–523.
8. ZHBANOV, Yu. K. and ZHURAVLEV, V. F., Metelitsyn's theorems. *J. Appl. Maths Mechs*, 2001, **65** 3, 525–527.
9. SCHMMEIDLER, W., *Vorträge über Determinanten und Matrizen mit Anwendungen in Physik und Technik*. Berlin: Akademie-Verlag, 1949.
10. MERKIN, D. R., *Introduction to Theory of Stability of Motion*. Nauka, Moscow, 1987.
11. MAGNUS, K., *Kreisel. Theorie und Anwendungen*. Springer, Berlin, 1971.
12. LANCASTER, P. and TISMENETSKY, M., *The Theory of Matrices*. Academic Press, San Diego, 1985.
13. MUELLER, P. C., *Stabilität und Matrizen*. Springer, Berlin, 1977.
14. ISHLINSKII, A. Yu., STOROZHENKO, V. A. and TEMCHENKO, M. E., *Investigation of the Stability of Complex Mechanical Systems*, Nauka, Moscow, 2002.
15. BOLOTIN, V. V., *Non-conservative Problems of the Theory of Elastic Stability*. Pergamon Press, Oxford, 1963.
16. GASCH, R., Stabilier Lauf von Turbinenrotoren. *Konstruktion*, 1965, **17**, 11, 447–452.

English version submitted by the authors